A Comparison Among Simple Algorithms for Linear Programming

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Problem

Consider the search for a feasible solution for the following set of linear restrictions:

$$\begin{aligned} Px &= 0, \\ e^t x &= 1, \\ x &\geq 0, \end{aligned} \tag{1}$$

where $P \in \Re^{m \times n}$, $x \in \Re^n$ and $e \in \Re^n$ and be the vector with all the coordinates equal to one and the P columns have norm one, i.e., $||Pj|| = 1$, for $j = 1, \dots, n$.

Note that any linear programming problem can be reduced to the problem (1).

von Neumann's Algorithm

Given: $x^0 \geq 0$, with $e^t x^0 = 1$. Compute $b^0 = Px^0$.

For $k = 1, 2, 3, \ldots$ **Do:**

1) Compute:

$$s = \text{argmin}_{j=1,\ldots,n}\{P_j^t b^{k-1}\},$$
$$v^{k-1} = P_s^t b^{k-1}.$$

2) If $v^{k-1} > 0$, then **STOP**. The problem (1) is infeasible.

3) Compute:

$$u^{k-1} = ||b^{k-1}||,$$
$$\lambda = \frac{1-v^{k-1}}{(u^{k-1})^2 - 2v^{k-1} + 1}.$$

4) Update:

$$b^k = \lambda b^{k-1} + (1-\lambda)P_s,$$
$$x^k = \lambda x^{k-1} + (1-\lambda)e_s,$$

End

von Neumann's Algorithm

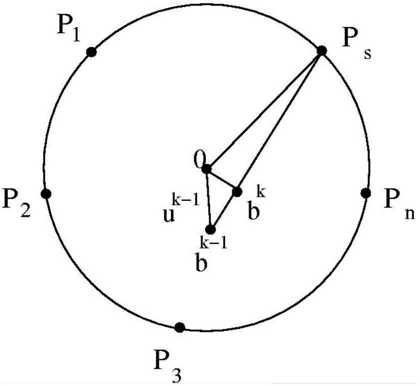

Illustration of the von Neumann's Algorithm.

Optimal Pair Adjustment Algorithm

Given: $x^0 \geq 0$, with $e^t x^0 = 1$. Compute $b^0 = Px^0$.

For $k = 1, 2, 3, \ldots$ **Do:**

1) Compute:

$$s^+ = \text{argmin}_{j=1,\ldots,n}\{P_j^t b^{k-1}\},$$
$$s^- = \text{argmax}_{j=1,\ldots,n}\{P_j^t b^{k-1} |\ x_j > 0\},$$
$$v^{k-1} = P_{s^+}^t b^{k-1}.$$

2) If $v^{k-1} > 0$, then **STOP**; the problem (1) is infeasible.

3) Solve the problem:

$$\begin{array}{ll} \text{minimize} & ||\overline{b}||^2 \\ \text{s.t.} & \lambda_0(1 - x_{s^+}^{k-1} - x_{s^-}^{k-1}) + \lambda_1 + \lambda_2 = 1, \\ & \lambda_i \geq 0, \ \text{for } i = 0, 1, 2. \end{array} \qquad (2)$$

where,

$$\overline{b} = \lambda_0(b^{k-1} - x_{s^+}^{k-1} P_{s^+} - x_{s^-}^{k-1} P_{s^-}) + \lambda_1 P_{s^+} + \lambda_2 P_{s^-}.$$

Optimal Pair Adjustment Algorithm

4) Update:

$$b^k = \lambda_0 (b^{k-1} - x_{s^+}^{k-1} P_{s^+} - x_{s^-}^{k-1} P_{s^-}) + \lambda_1 P_{s^+} + \lambda_2 P_{s^-},$$

$$x_j^k = \begin{cases} \lambda_0 x_j^{k-1}, & j \neq s^+ \text{ e } j \neq s^-, \\ \lambda_1, & j = s^+, \\ \lambda_2, & j = s^-. \end{cases}$$

End.

Optimal Adjustment Algorithms for p Coordinates

Given: $x^0 \geq 0$, with $e^t x^0 = 1$. Compute $b^0 = Px^0$.
For $k = 1, 2, 3, \ldots$ **Do:**
1) Compute:

$\{P_{\eta_1^+}, \ldots, P_{\eta_{s_1}^+}\}$ forming the largest angle with the vector b^{k-1}.
$\{P_{\eta_1^-}, \ldots, P_{\eta_{s_2}^-}\}$ forming the smallest angle with the vector b^{k-1}
such as $x_i^{k-1} > 0, i = \eta_1^-, \ldots, \eta_{s_2}^-$, where $s_1 + s_2 = p$.

$v^{k-1} = \text{minimum}_{i=1,\ldots,s_1}\{P^t_{\eta_i^+} b^{k-1}\}$.
2) If $v^{k-1} > 0$, then **STOP**; the problem (1) is infeasible.

Optimal Adjustment Algorithms for p Coordinates

3) Solve the problem:

$$\begin{array}{ll} \text{minimize} & ||\overline{b}||^2 \\ \text{s.t.} & \lambda_0 \left(1 - \sum_{i=1}^{s_1} x_{\eta_i^+}^{k-1} - \sum_{j=1}^{s_2} x_{\eta_j^-}^{k-1}\right) + \sum_{i=1}^{s_1} \lambda_{\eta_i^+} + \sum_{j=1}^{s_2} \lambda_{\eta_j^-} = 1, \\ & \lambda_{\eta_i^+} \geq 0, \quad \text{for } i = 1, \ldots, s_1, \\ & \lambda_{\eta_j^-} \geq 0, \quad \text{for } j = 1, \ldots, s_2, \end{array} \tag{3}$$

where

$$\overline{b} = \lambda_0 \left(b^{k-1} - \sum_{i=1}^{s_1} x_{\eta_i^+}^{k-1} P_{\eta_i^+} - \sum_{j=1}^{s_2} x_{\eta_j^-}^{k-1} P_{\eta_j^-}\right) + \sum_{i=1}^{s_1} \lambda_{\eta_i^+} P_{\eta_i^+} + \sum_{j=1}^{s_2} \lambda_{\eta_j^-} P_{\eta_j^-}.$$

Optimal Adjustment Algorithms for p Coordinates

4) Update:

$$b^k = \lambda_0 \left(b^{k-1} - \sum_{i=1}^{s_1} x^{k-1}_{\eta_i^+} P_{\eta_i^+} - \sum_{j=1}^{s_2} x^{k-1}_{\eta_j^-} P_{\eta_j^-} \right) + \sum_{i=1}^{s_1} \lambda_{\eta_i^+} P_{\eta_i^+} + \sum_{j=1}^{s_2} \lambda_{\eta_j^-} P_{\eta_j^-},$$

$$x_j^k = \begin{cases} \lambda_0 x_j^{k-1}, & j \notin \{\eta_1^+, \ldots, \eta_{s_1}^+, \eta_1^-, \ldots, \eta_{s_2}^-\}, \\ \lambda_{\eta_i^+}, & j = \eta_i^+, i = 1, \ldots, s_1, \\ \lambda_{\eta_j^-}, & j = \eta_j^-, j = 1, \ldots, s_2. \end{cases}$$

End

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- 2) **If** $v^{k-1} > 0$, **then STOP; the problem (1) is infeasible.**

Equivalence between algorithms with $p = 1$ and von Neumann

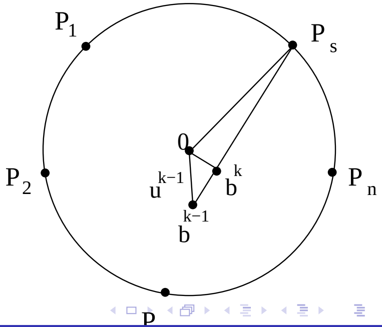
- **They differ in the Step 3). In the von Neumann's algorithm.**

We solve the following subproblem

$$\begin{array}{ll} \text{minimize} & ||\overline{b}||^2 \\ \text{s.t.} & \lambda \in [0, 1] \end{array}$$

where, $\overline{b} = \lambda b^{k-1} + (1 - \lambda) P_s$.

This is done by computing the optimal λ.



Equivalence between algorithms with $p = 1$ and von Neumann

In the algorithm with $p = 1$,

we have to solve the subproblem.

$$\begin{aligned} &\text{minimize} \quad ||\overline{b}||^2 \\ &\text{s.t.} \quad \lambda_0(1 - x_s^{k-1}) + \lambda_1 = 1, \\ &\lambda_i \geq 0, \text{ for } i = 0, 1. \end{aligned}$$

where,

$$\overline{b} = \lambda_0(b^{k-1} - x_s^{k-1} P_s) + \lambda_1 P_s.$$

We can rewrite the subproblem as follows:

$$\lambda_0(1 - x_s^{k-1}) + \lambda_1 = 1 \Leftrightarrow$$

$$\lambda_1 = 1 - \lambda_0(1 - x_s^{k-1}) \geq 0$$

$$\begin{aligned} \overline{b} &= \lambda_0(b^{k-1} - x_s^{k-1} P_s) + \lambda_1 P_s \\ &= \lambda_0(b^{k-1} - x_s^{k-1} P_s) + \\ &\quad (1 - \lambda_0(1 - x_s^{k-1})) P_s \\ &= \lambda_0 b^{k-1} + (1 - \lambda_0) P_s. \end{aligned}$$

and

$$0 \leq \lambda_0 \leq \frac{1}{1 - x_s^{k-1}}$$

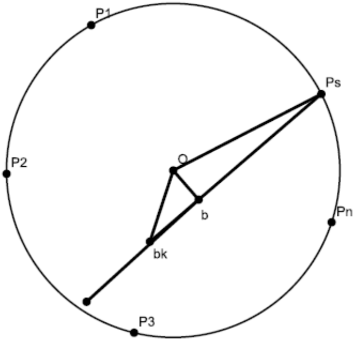
Equivalence between algorithms with $p = 1$ and von Neumann

This, we have the problem

$$\begin{array}{ll} \text{minimize} & ||\overline{b}||^2 \\ \text{s.t.} & \lambda \in \left[0, \frac{1}{1-x_s^{k-1}}\right] \end{array}$$

where, $\overline{b} = \lambda b^{k-1} + (1-\lambda)P_s.$

As $\frac{1}{1-x_s^{k-1}} > 1$, so the geometric view of algorithm with $p = 1$ is given by Figure 4.



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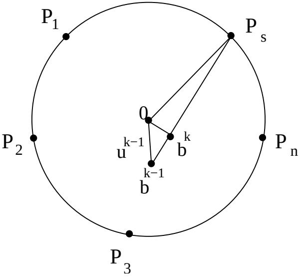


Figura: Illustration of the von Neumann's algorithm

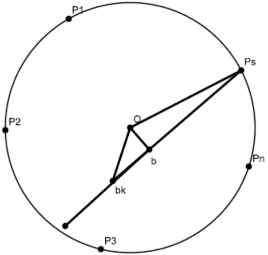


Figura: Illustration of the algorithm with $p = 1$.

Equivalence between algorithms with $p = 1$ and von Neumann

In the Step 4) of the von Neumann's algorithm x^k is updated by

$$x_j^k = \begin{cases} \lambda x_j^{k-1}, & j \neq s \\ \lambda(x_s^{k-1} - 1) + 1, & j = s. \end{cases}$$

In the Step 4) of the algorithm with $p = 1$, x^k is updated by

$$x_j^k = \begin{cases} \lambda x_j^{k-1}, & j \neq s \\ 1 - \lambda(1 - x_s^{k-1}), & j = s. \end{cases}$$

Geometric view of the Optimal Adjustment Algorithms for p Coordinates

For the case $p = 2$, the subproblem of Step 3) reduces to the following form:

$$\begin{array}{ll} \text{minimize} & ||\overline{b}||^2 \\ \text{s.t.} & \lambda_0(1 - x_{s^+}^{k-1} - x_{s^-}^{k-1}) + \lambda_1 + \lambda_2 = 1, \quad \text{where,} \\ & \lambda_i \geq 0, \text{ for } i = 0, 1, 2. \end{array}$$

$$\overline{b} = \lambda_0(b^{k-1} - x_{s^+}^{k-1} P_{s^+} - x_{s^-}^{k-1} P_{s^-}) + \lambda_1 P_{s^+} + \lambda_2 P_{s^-}.$$

We can rewrite $\overline{b}$ as:

$$\begin{aligned} \overline{b} &= \lambda_0(b^{k-1} - x_{s^+}^{k-1} P_{s^+} - x_{s^-}^{k-1} P_{s^-}) + \lambda_1 P_{s^+} + \lambda_2 P_{s^-} \\ &= \lambda_0 b^{k-1} + (\lambda_1 - \lambda_0 x_{s^+}^{k-1}) P_{s^+} + (\lambda_2 - \lambda_0 x_{s^-}^{k-1}) P_{s^-} \end{aligned}$$

and as we have $\lambda_0 + (\lambda_1 - \lambda_0 x_{s^+}^{k-1}) + (\lambda_2 - \lambda_0 x_{s^-}^{k-1}) = 1$, then $\overline{b}$ is an affine combination.

Geometric view of the Optimal Adjustment Algorithms for p Coordinates

- **The set formed by all combinations is the plane containing the points b^{k-1}, P_{s^+} and P_{s^-}.**

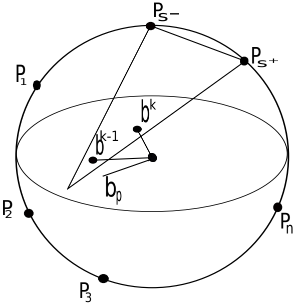


Figura: Illustration of the algorithm with $p = 2$.

Geometric view of the Optimal Adjustment Algorithms for p Coordinates

- The set formed by all combinations is the plane containing the points b^{k-1}, P_{s^+} and P_{s^-}.
- $\overline{b}(\lambda_0, \lambda_1, \lambda_2)$ **is a linear transformation with domain formed by all points of the restriction** $\lambda_0(1-x_{s^+}^{k-1}-x_{s^-}^{k-1})+\lambda_1+\lambda_2 = 1$

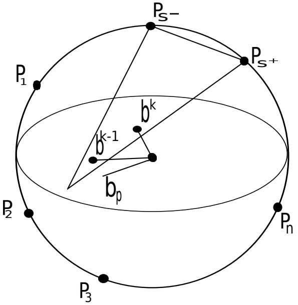
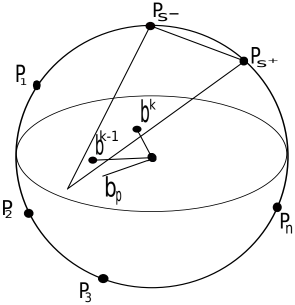


Figura: Illustration of the algorithm with $p = 2$.

Geometric view of the Optimal Adjustment Algorithms for p Coordinates

- **The image this set by linear transformation is the triangle with vertices in P_{s^+}, P_{s^-} and P_v and its interior in the affine space, where** $P_v = \frac{1}{(1-x_{s^+}^{k-1}-x_{s^-}^{k-1})}(b^{k-1} - x_{s^+}^{k-1}P_{s^+} - x_{s^-}^{k-1}P_{s^-})$



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Geometric view of the Optimal Adjustment Algorithms for p Coordinates

- The image this set by linear transformation is the triangle with vertices in P_{s^+}, P_{s^-} and P_v and its interior in the affine space, where $P_v = \frac{1}{(1-x_{s^+}^{k-1}-x_{s^-}^{k-1})}(b^{k-1} - x_{s^+}^{k-1}P_{s^+} - x_{s^-}^{k-1}P_{s^-})$
- **The optimal residue b^k that we seek is the projection of the origin on this triangle**

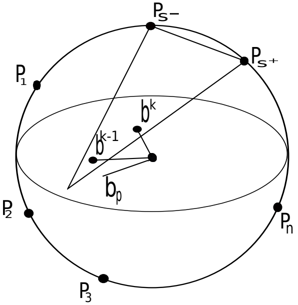


Figura: Illustration of the algorithm with $p = 2$.

Geometric view of the Optimal Adjustment Algorithms for p Coordinates

- **In the case of the algorithm for p coordinates**

Geometric view of the Optimal Adjustment Algorithms for p Coordinates

- In the case of the algorithm for p coordinates
- **The optimal residue b^k is the projection of the origin on the affine space region with vertices in the p columns and in the vector P_v.**

Subproblem Solution Using Interior Points Methods

First, we rewrite the sub-problem in the matrix format:

$$\begin{aligned} &\text{minimize } \tfrac{1}{2}||W\lambda||^2 \\ &\text{s.a } \; c^t\lambda = 1, \\ &\qquad -\lambda \leq 0, \end{aligned} \tag{4}$$

$$\begin{aligned} W &= \left[\overline{w}\; P_{\eta_1^+} \dots P_{\eta_{s_1}^+} P_{\eta_1^-} \dots P_{\eta_{s_2}^-}\right], \\ \overline{w} &= b^{k-1} - \sum_{i=1}^{s_1} x_{\eta_i^+}^{k-1} P_{\eta_i^+} - \sum_{j=1}^{s_2} x_{\eta_j^-}^{k-1} P_{\eta_j^-}, \\ \lambda &= \left(\lambda_0, \lambda_{\eta_1^+}, \dots, \lambda_{\eta_{s_1}^+}, \dots, \lambda_{\eta_1^-}, \dots, \lambda_{\eta_{s_2}^-}\right), \\ c &= (c_1, 1, \dots, 1), \; c_1 = 1 - \sum_{i=1}^{s_1} x_{\eta_i^+}^{k-1} - \sum_{j=1}^{s_2} x_{\eta_j^-}^{k-1}. \end{aligned} \tag{5}$$

Subproblem Solution Using I nterior Points Methods

The KKT equations from the problem (4) are given by:

$$\begin{aligned} W^tW\lambda + cl - \mu &= 0 \\ \mu^t\lambda &= 0 \\ c^t\lambda - 1 &= 0, \end{aligned} \qquad (6)$$

where τ is free, $0 \leq \mu$ are the Lagrange multipliers for equality and inequality, respectively, and the W^tW matrix is of the order $(p+1) \times (p+1)$.
Next, we apply the path-following interior point method to problem (6).

Subproblem Solution Using Interior Points Methods

The linear system that arises at the iteration of the interior point method applied to these equations has the form:

$$\begin{pmatrix} W^tW & c & -Id \\ U & 0 & \Lambda \\ c^t & 0 & 0 \end{pmatrix} \begin{bmatrix} d\lambda \\ dl \\ d\mu \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad (7)$$

where

$$\begin{aligned} U &= diag(\mu), \\ \Lambda &= diag(\lambda), \\ r_1 &= \mu - cl - W^tW\lambda, \\ r_2 &= -l^t\lambda, \\ r_3 &= 1 - c^t\lambda. \end{aligned}$$

Subproblem Solution Using Interior Points Methods

$$d\mu = \Lambda^{-1} r_2 - \Lambda^{-1} U d\lambda,$$
$$d\lambda = (W^t W + \Lambda^{-1} U)^{-1} r_4 - (W^t W + \Lambda^{-1} U)^{-1} c\, dl,$$
$$c^t (W^t W + \Lambda^{-1} U)^{-1} c\, dl = c^t (W^t W + \Lambda^{-1} U)^{-1} r_4 - r_3,$$

$$(W^t W + \Lambda^{-1} U) sl1 = c$$
$$(W^t W + \Lambda^{-1} U) sl2 = r_4$$
$$r_4 = r_1 + \Lambda^{-1} r_2$$

Theoretical Properties of the New Method

Theorem

The decrease in $||b^k||$ obtained by an iteration of the optimal adjustment algorithm for p coordinates, with $1 \leq p \leq n$, where n is the number of P columns, in the worst scenario, is equal to the one obtained by an iteration of the von Neumann's algorithm

Theoretical Properties of the New Method

Theorem

The decrease in $||b^k||$ obtained by an iteration of the optimal adjustment algorithm for p_2 coordinates, in the worst scenario, is equal to the one obtained by an iteration of the optimal adjustment algorithm for p_1 coordinates with $p_1 \leq p_2 \leq n$, where n is the number of columns P.

Sufficient Condition for $||b^k|| < ||b^k_v||$

- Let b^k be the residue of the algorithm with $p = 2$ in the iteration k and let P_{s^+} and P_{s^-} be the columns forming the largest and smallest angles with the vector b^{k-1}.

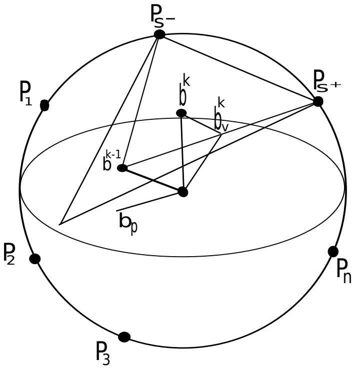
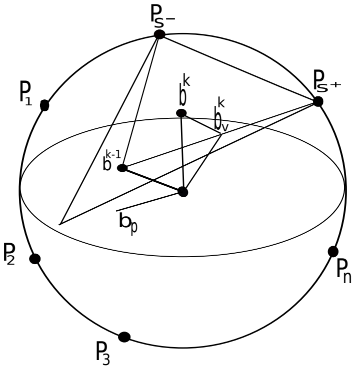


Figura: Illustration of the Sufficient Condition.

Sufficient Condition for $||b^k|| < ||b_v^k||$

- Let b^k be the residue of the algorithm with $p = 2$ in the iteration k and let P_{s^+} and P_{s^-} be the columns forming the largest and smallest angles with the vector b^{k-1}.
- If the projection of the origin is in the interior of the triangle $b^k P_{s^+} P_{s^-}$



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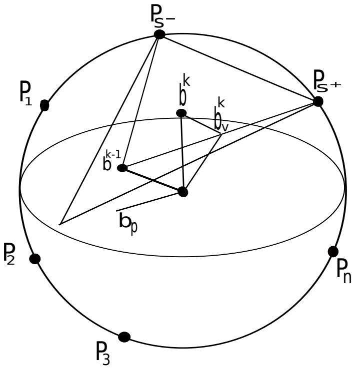
Sufficient Condition for $||b^k|| < ||b_v^k||$

- **And coincides with the projection of the origin in the plane determined by b^{k-1}, P_{s^+} and P_{s^-}, then $||b^k|| < ||b_v^k||$, where b_v^k is the residue of the von Neumann's algorithm in the iteration k.**



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- **We can see in the Figure 8 that the triangle $0b^k b_v^k$ has the $\overline{0b_v^k}$ hypotenuse and side $\overline{0b^k}$.**

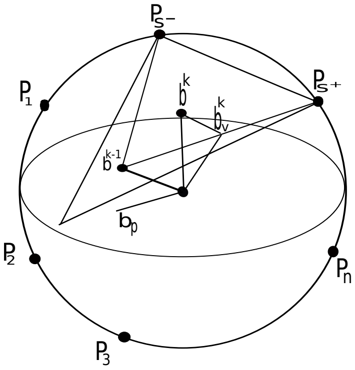

Figura: Illustration of the Sufficient Condition.

Computational Experiments

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- We used a collection of 151 linear programming problems.
- **The problems are divided into 95 Netlib problems, 16 Kennington problems, and 40 other problems, which are not publicly available, and were supplied by Gonçalves [5]**

Computational Experiments

- **The family of algorithms was implemented in C.**

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- **To solve the subproblem, the classical path-following interior point method was implemented in C**

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3. **Next, the times $t2$, $t3$, $t4$ and $t5$ (CPU seconds), which respectively correspond to 3, 5, 10 and 20 times the number of iterations in $t1$ are also recorded.**

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4. **After that, the optimal adjustment algorithm for p coordinates where $p = 2$, $p = 4$, $p = 10$ and $p = 20$ is ran on the test problems.**

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4. After that, the optimal adjustment algorithm for p coordinates where $p = 2$, $p = 4$, $p = 10$ and $p = 20$ is ran on the test problems.
5. **Finally, for the ti times, $i = 1, \ldots, 5$, the residue $||b^k||$ is recorded.**

Computational Results

Tabela: Percentage of the relative gain by the algorithms on the problems in five different times

Algorithm	t1	t2	t3	t4	t5
Algorithm with p=2	15,89%	11,92%	22,51%	5,96%	7,94 %
Algorithm with p=4	31,12%	34,43%	27,81%	19,96%	23,83 %
Algorithm with p=10	20,52%	27,15%	25,16%	25,16%	23,84 %
Algorithm with p=20	32,45%	26,49%	24,50%	50,99%	44,37%

Computational Results

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- **The distance of the residue $||b^k||$ to the origin, was used to measure the performance.**

Computational Results

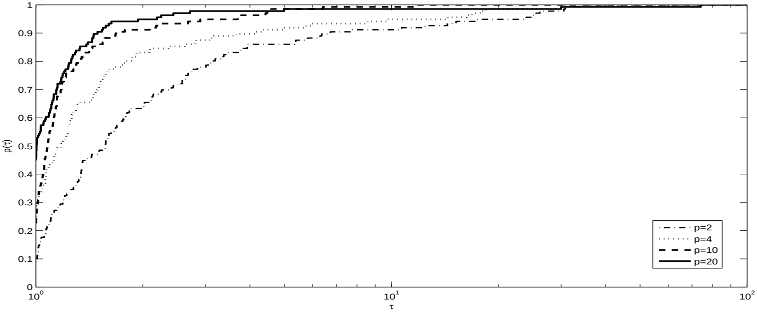

Figura: Profile Performance of four algorithms in t5 time.

Computational Results

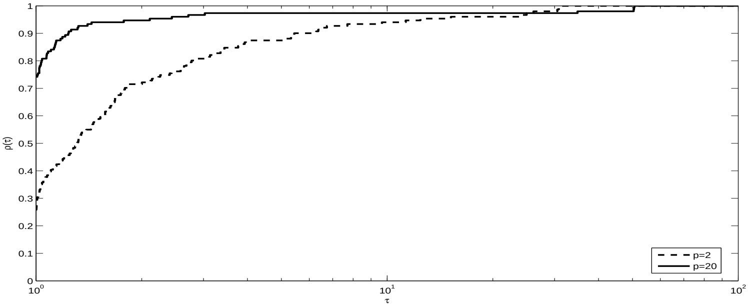

Figura: Performance profile of the algorithms with $p = 2$ and $p = 20$ in time $t5$.

Computational Results

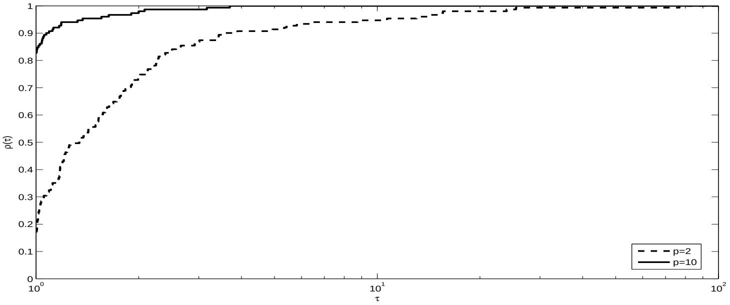

Figura: Performance profile of the algorithms with $p = 2$ and $p = 10$ in time $t5$.

Computational Results

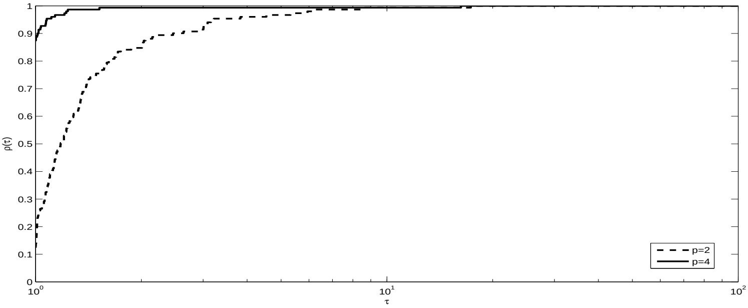

Figura: Performance profile of the algorithms with $p = 2$ and $p = 4$ in time $t5$.

Thank You

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